Regular Separability of WSTS

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Given $\mathcal{L}, \mathcal{K} \subseteq \Sigma^*$ from class $\mathcal{F}$.
What is their relationship?

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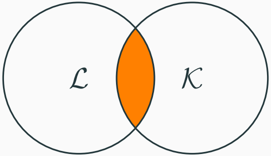
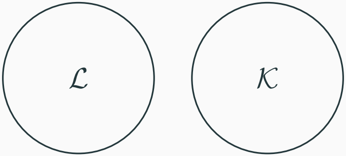
Case 1: $\mathcal{L} \cap \mathcal{K} \neq \varnothing$



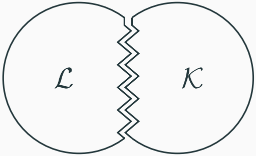
↳ study $\mathcal{L} \cap \mathcal{K}$

Separability

Case 2: $\mathcal{L} \cap \mathcal{K} = \varnothing$



vs.



Separability

Consider separability

<u>Separability of $\mathcal{F}$ by $\mathcal{S}$</u>

Given: Languages $\mathcal{L}, \mathcal{K} \subseteq \Sigma^*$ from $\mathcal{F}$

Decide: Is there $\mathcal{R} \subseteq \Sigma^*$ from $\mathcal{S}$ such that

$$\mathcal{L} \subseteq \mathcal{R}, \quad \mathcal{K} \cap \mathcal{R} = \emptyset?$$

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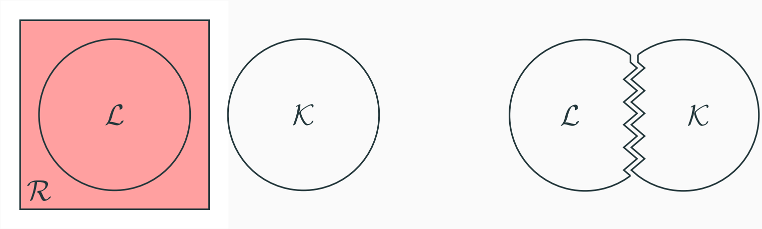

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Commonly studied:

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  ↳ Separability is decidable [PZ16]

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- $\mathcal{S} = REG \subset \mathcal{F}$
  Regular separability
  (related work in a second)

Regular separability

Regular separability of $\mathcal{F}$

Given: Languages $\mathcal{L}, \mathcal{K} \subseteq \Sigma^*$ from $\mathcal{F}$

Decide: Is there $\mathcal{R} \subseteq \Sigma^*$ regular such that

$$\mathcal{L} \subseteq \mathcal{R}, \quad \mathcal{K} \cap \mathcal{R} = \varnothing?$$

Observation:

Problem is symmetric in the input:

If $\quad \mathcal{L} \subseteq \mathcal{R}, \quad \mathcal{K} \cap \mathcal{R} = \varnothing,$

then $\quad \mathcal{K} \subseteq \overline{\mathcal{R}}, \quad \mathcal{L} \cap \overline{\mathcal{R}} = \varnothing.$

↳ Call $\mathcal{L}, \mathcal{K}$ regularly separable if separator $\mathcal{R}$ exists.

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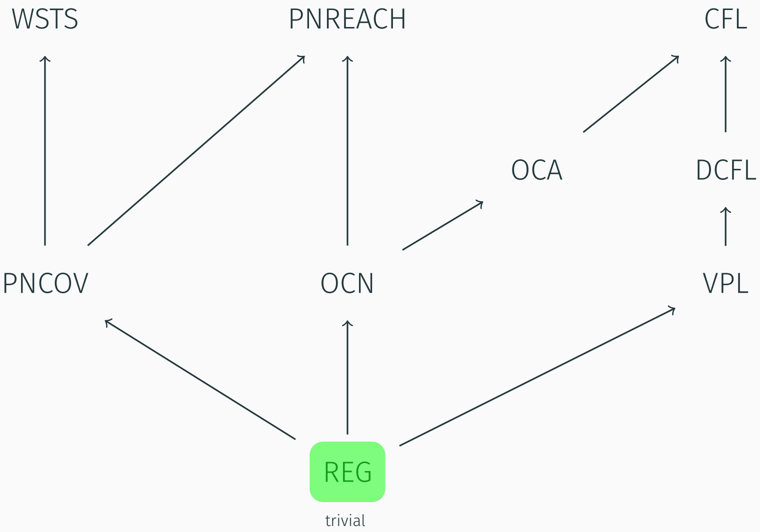
Disjointness is always a necessary condition for any kind of separability.

It is not always sufficient, consider

$$\mathcal{L} = a^n b^n, \quad \mathcal{K} = \overline{\mathcal{L}}\ .$$

Regular separability - A map



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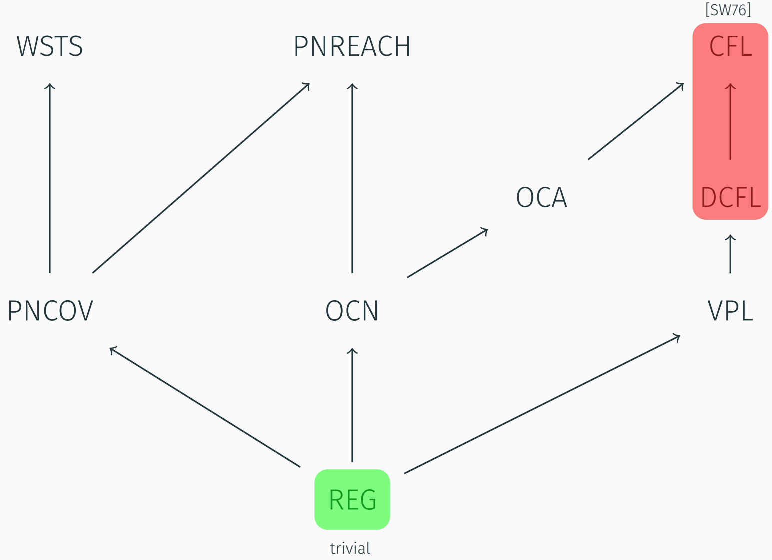

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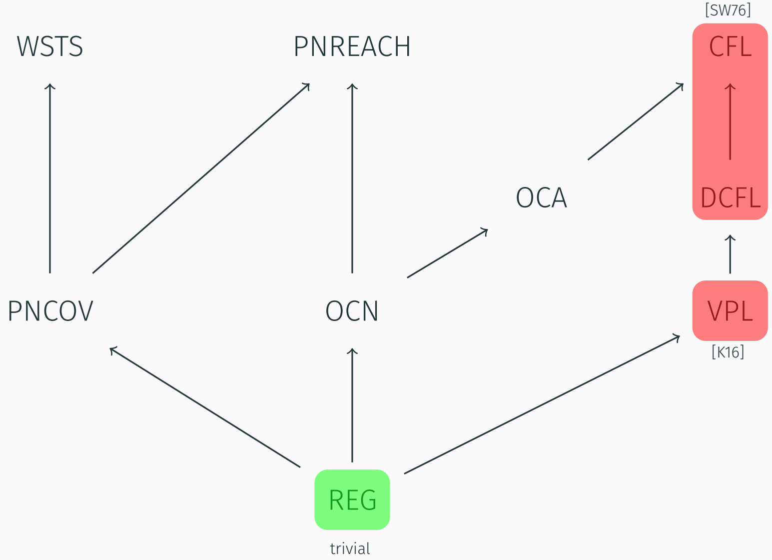

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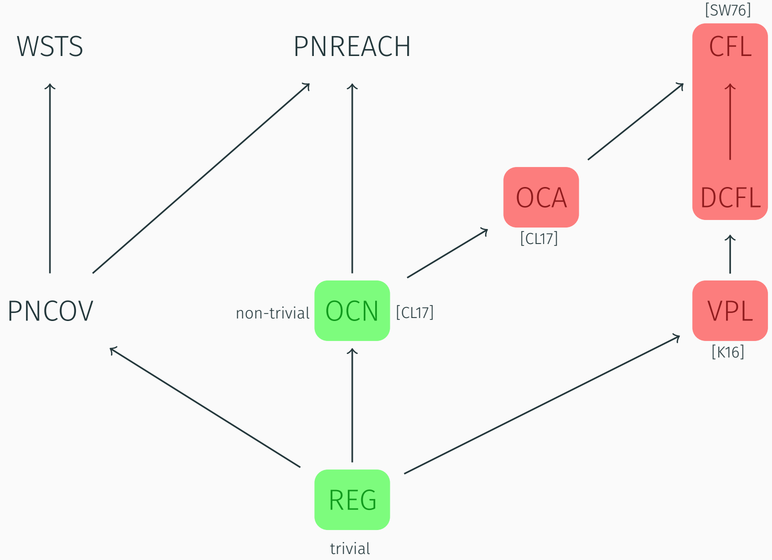

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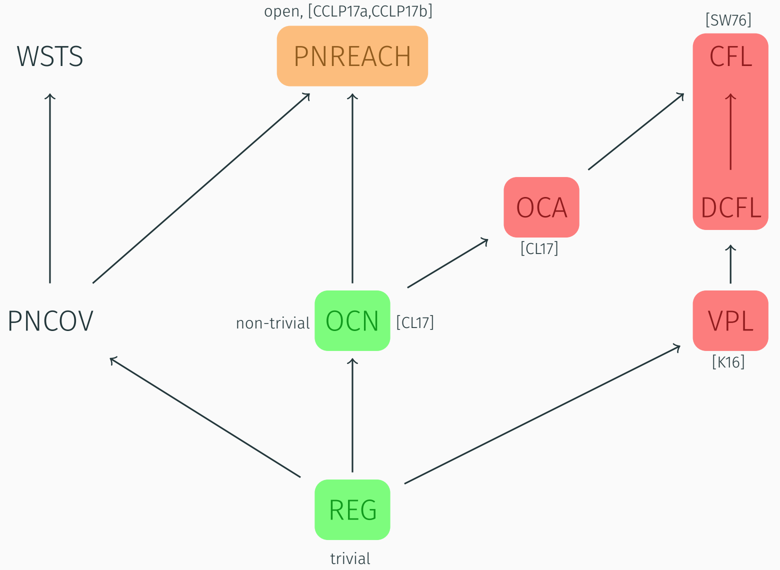

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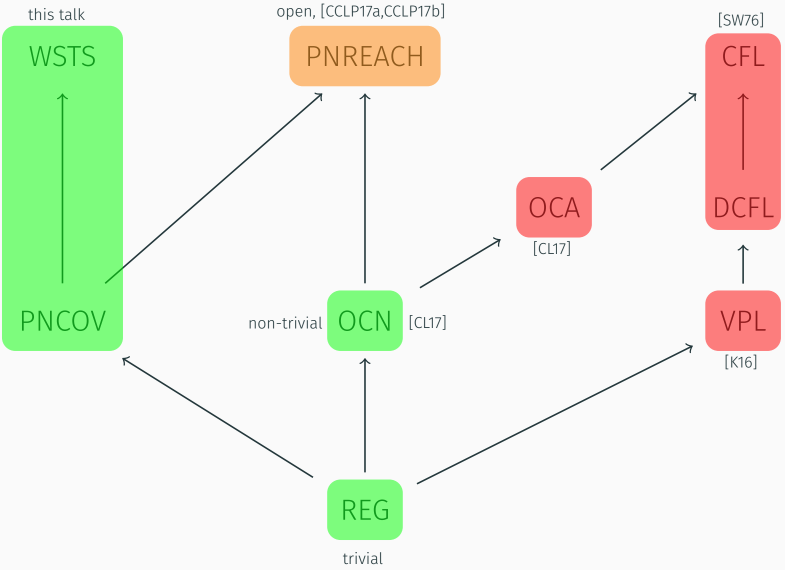

Well-structured transition systems

Well quasi orders

Consider $(X, \leqslant)$ quasi order (reflexive, transitive)

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Lemma (Dickson's lemma)

$(\mathbb{N}^k, \leqslant^k)$ *is a well quasi order*

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Lemma (Higman's lemma)

$(\Sigma^*, \leqslant^*)$ *is a well quasi order*

RADAR $\leqslant^*$ ABRACADABRA

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$T \subseteq S \times \Sigma \times S$ labeled transitions

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$F \subseteq S$ final states, upward-closed

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- $T \subseteq S \times \Sigma \times S$ labeled transitions
- $I \subseteq S$ initial states
- $F \subseteq S$ final states, upward-closed
- Monotonicity / Simulation property:

$$\begin{array}{ccc} s' & \overset{a}{\dashrightarrow} & r' \ (\exists) \\ \rotatebox{90}{$\leqslant$} & & \rotatebox{90}{$\leqslant$} \\ s & \xrightarrow{a} & r \end{array}$$

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Coverability language

$$\mathcal{L}(\mathcal{W}) = \left\{ w \in \Sigma^* \;\middle|\; c_i \xrightarrow{w} c_f \text{ for some } c_i \in I, c_f \in F \right\}$$

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Example 1:

Labeled Petri net with covering M_f as acceptance condition induces WSTS

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Example 2:

Labeled lossy channel system (LCS) [AJ93] induces a WSTS.

The result & and its consequences

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Theorem

If two WSTS languages, one of them finitely branching, are disjoint, then they are regularly separable.

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Corollary

No subclass of finitely-branching WSTS beyond REG is closed under complement.

Expressibility results

Our result - Recall

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How much of a restriction is it to assume finite branching?

What do we gain by assuming finite branching?

Expressibility I

Proposition

Languages of ω^2-WSTS

$\subseteq$ Languages of finitely branching WSTS.

$$\begin{array}{ll} & (S, \leqslant)\ \omega^2 \text{ wqo} \\ \text{iff} & (\mathcal{P}^{\downarrow}(S), \subseteq) \text{ wqo} \\ \text{iff} & (S, \leqslant) \text{ does not embed the Rado order} \end{array}$$

Our result applies to all WSTS of practical interest!

Expressibility II

Proposition

Languages of finitely branching WSTS

= Languages of deterministic WSTS.

Sufficient to show:

Theorem

If two WSTS languages, one of them deterministic, are disjoint, then they are regularly separable.

Proof sketch

Proof approach

Theorem

If two WSTS languages, one of them deterministic, are disjoint, then they are regularly separable.

Proof approach:

Relate separability to the existence of certain invariants:

> Separability talks about the languages,
>
> Invariants talk about the state space!

Inductive invariant

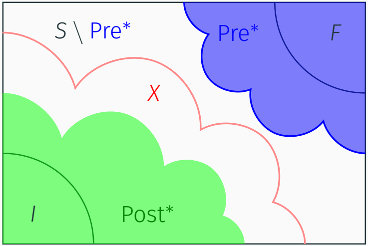
Inductive invariant [MP95] X for WSTS $\mathcal{W}$:

(1) $X \subseteq S$ downward-closed

(2) $I \subseteq X$

(3) $F \cap X = \varnothing$

(4) $\mathrm{Post}_{\Sigma}(X) \subseteq X$



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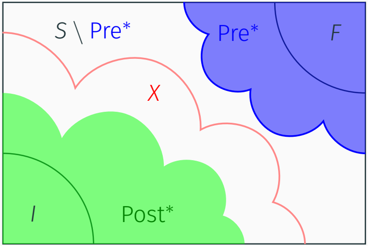

Lemma

$\mathcal{L}(\mathcal{W}) = \varnothing$ iff inductive invariant for $\mathcal{W}$ exists.

Proof approach

$$\mathcal{L}(\mathcal{W}_1), \mathcal{L}(\mathcal{W}_2) \text{ reg. sep} \overset{!}{\iff} \mathcal{L}(\mathcal{W}_1) \cap \mathcal{L}(\mathcal{W}_2) = \mathcal{L}(\mathcal{W}_1 \times \mathcal{W}_2) = \varnothing$$

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$$\Updownarrow$$

$\mathcal{W}_1 \times \mathcal{W}_2$ has inductive invariant

Proof approach

$$\begin{tikzcd}
\mathcal{L}(\mathcal{W}_1), \mathcal{L}(\mathcal{W}_2) \text{ reg. sep} \arrow[rr, Leftrightarrow, "!"] & & \mathcal{L}(\mathcal{W}_1) \cap \mathcal{L}(\mathcal{W}_2) = \mathcal{L}(\mathcal{W}_1 \times \mathcal{W}_2) = \varnothing \\
& \mathcal{W}_1 \times \mathcal{W}_2 \text{ has inductive invariant} \arrow[ul, Rightarrow, "?"] \arrow[ur, Leftrightarrow] &
\end{tikzcd}$$

Finitely represented invariants

The desired implication does not hold.

Call an invariant X finitely represented if $X = Q\downarrow$ for Q finite

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Recall:

$(S, \leqslant)$ well quasi order (wqo)
iff upward-closed sets have finitely many minimal elements.

No such statement for downward-closed sets and maximal elements!

Finitely represented invariants

The desired implication does not hold.

Call an invariant X finitely represented if $X = Q\downarrow$ for Q finite

We can show:

Theorem

Let $\mathcal{W}_1, \mathcal{W}_2$ WSTS, $\mathcal{W}_2$ deterministic.

If $\mathcal{W}_1 \times \mathcal{W}_2$ admits a finitely-represented inductive invariant, then $\mathcal{L}(\mathcal{W}_1)$ and $\mathcal{L}(\mathcal{W}_2)$ are regularly separable.

Proof approach II

$$
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& \mathcal{W}_1 \times \mathcal{W}_2 \text{ has fin.-rep. invariant} \arrow[ul, Rightarrow, "\checkmark"'] &
\end{tikzcd}
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✓ ✗

$\mathcal{W}_1 \times \mathcal{W}_2$ has fin.-rep. invariant

Ideals

Finitely represented invariants do not necessarily exist.

Solution: Ideals

Definition

For WSTS $\mathcal{W}$, let $\widehat{\mathcal{W}}$ be its ideal completion. [KP92][BFM14,FG12]

Lemma

$\mathcal{L}(\mathcal{W}) = \mathcal{L}(\widehat{\mathcal{W}})$.

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Proposition

If X is an inductive invariant for $\mathcal{W}$,
then its ideal decomposition $\textsc{Idec}(X)\downarrow$
is a finitely-represented inductive invariant for $\widehat{\mathcal{W}}$.

Proof

Putting everything together:

If $\mathcal{W}_1, \mathcal{W}_2$ are disjoint, $\mathcal{W}_1 \times \mathcal{W}_2$ admits an invariant X.

Then $\textsc{Idec}(X)\downarrow$ is a finitely-represented invariant for $\widehat{\mathcal{W}_1 \times \mathcal{W}_2} \cong \widehat{\mathcal{W}_1} \times \widehat{\mathcal{W}_2}$.

This finitely-represented invariant gives rise to a regular separator.

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We have shown:

Theorem

If two WSTS languages are disjoint,
one of them finitely branching or deterministic or ω^2,
then they are regularly separable.

Proof details:

From fin.-rep. invariants to regular separators

From invariants to separability

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Assume $Q{\downarrow}$ is invariant.

Idea: Construct separating NFA with Q as states

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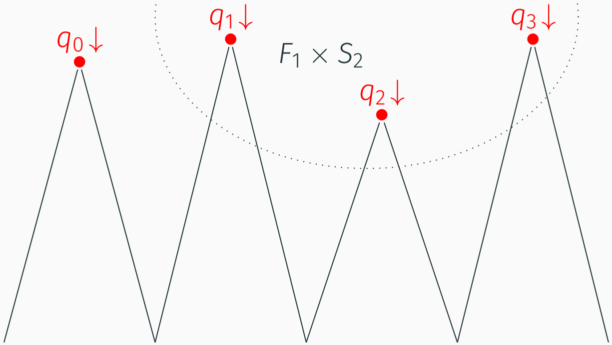
$$\begin{tikzcd}
 & (r, r') \in Q \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\leqslant$}"] \\
Q \ni (s, s') \arrow[ur, bend left, "a", "\text{in } \mathcal{A}"'] \arrow[r, "a", "\text{in } \mathcal{W}_1 \times \mathcal{W}_2"'] & (t, t') \in S_1 \times S_2
\end{tikzcd}$$

Behavior of $\mathcal{A}$



$\mathcal{A}$ over-approximates the behavior of the product system using the configurations from Q.

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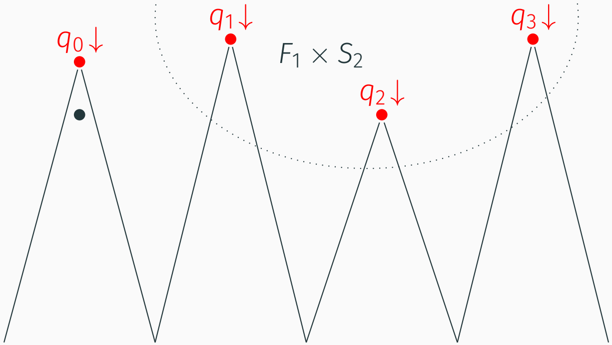

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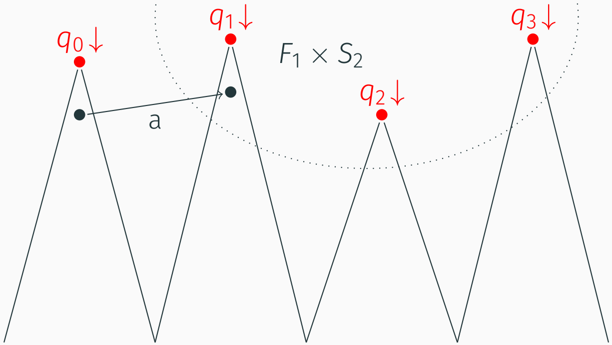

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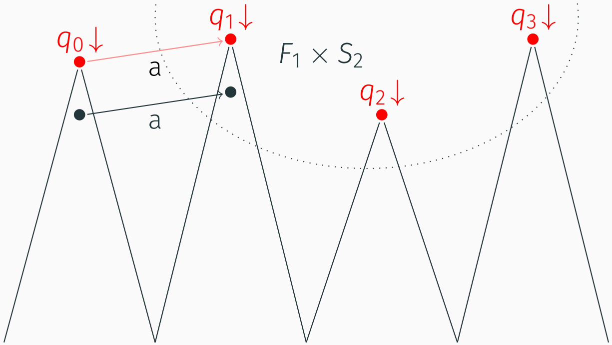

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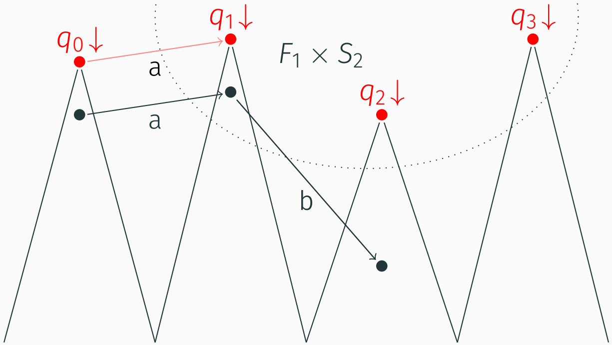

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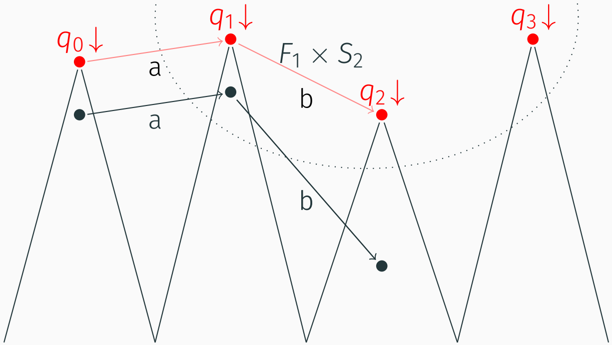

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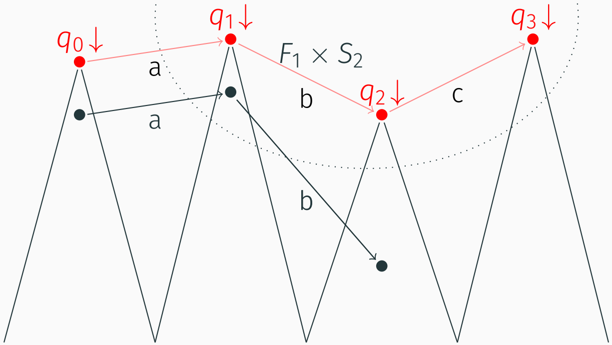

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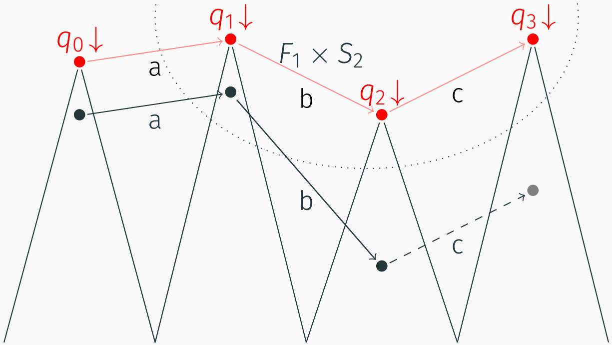

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Proving separability: Inclusion

Lemma

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Proof.

Any run $c \xrightarrow{w} d$ of $\mathcal{W}_1$

synchronizes with **the** run of $\mathcal{W}_2$ for w

in the run $(c, c') \xrightarrow{w} (d, d')$ of $\mathcal{W}_1 \times \mathcal{W}_2$.

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Proving separability: Inclusion

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$\mathcal{L}(\mathcal{W}_1) \subseteq \mathcal{L}(\mathcal{A})$.

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If d is final in $\mathcal{W}_1$,

the over-approximation of (d, d') is final in $\mathcal{A}$. $\square$

Proving separability: Disjointness

Lemma

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Proof.

Any run of $\mathcal{A}$ for w over-approximates

in the second component the unique run of $\mathcal{W}_2$ for w.

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If $w \in \mathcal{L}(\mathcal{W}_2) \cap \mathcal{L}(\mathcal{A})$

then some run of $\mathcal{A}$ reaches a state (q, q') with

- q final in $\mathcal{W}_1$ (def. of Q_I)
- q' final in $\mathcal{W}_2$ ($w \in \mathcal{L}(\mathcal{W}_2)$ + argument above)

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Contradiction to $F_1 \times F_2 \cap Q\downarrow = \varnothing$! $\square$

Proof details:
The ideal completion and fin.-rep. invariants

Finitely represented invariants

Lemma

Let $U \subseteq S$ be an upward-closed set in a wqo.

There is a finite set U_{min} such that $U = U_{min}\uparrow$.

A similar result for downward-closed subsets and maximal elements does not hold.

Finitely represented invariants

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Example:

Consider $\mathbb{N}$ in $(\mathbb{N}, \leqslant)$

Intuitively, $\mathbb{N} = \omega\downarrow$

Finitely represented invariants

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A similar result for downward-closed subsets and maximal elements does not hold.

Consequence:
Finitely represented invariants may not exist!

Solution:
Move to a language-equivalent system for which they always exist.

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Let $(S, \leqslant)$ be a wqo
An ideal $\mathcal{I} \subseteq S$ is a set that is

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Example 1:

For each $c \in S$, $c\!\downarrow$ is an ideal

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Example 2:
Consider $(\mathbb{N}^k, \leqslant)$
The ideals are the sets $u{\downarrow}$ for $u \in (\mathbb{N} \cup \{\omega\})^k$

Ideal decomposition

Lemma ([KP92])

Let $(S, \leqslant)$ be a wqo

For $D \subseteq S$ downward closed, let $\textsc{Idec}(D)$ be the set of inclusion-maximal ideals in D

$\textsc{Idec}(D)$ is unique, finite and we have

$$D = \bigcup \textsc{Idec}(D)$$

Ideal completion

Definition ([BFM14,FG12])

Let $\mathcal{W} = (S, \leqslant, T, I, F)$ WSTS

Its ideal completion is
$\widehat{\mathcal{W}} = (\{\mathcal{I} \subseteq S \mid \mathcal{I} \text{ ideal}\}, \subseteq, \widehat{T}, \textsc{Idec}(I{\downarrow}), \widehat{F})$ with

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Lemma

- $\widehat{\mathcal{W}}$ *finitely branching*
- $\mathcal{W}$ *deterministic* $\implies$ $\widehat{\mathcal{W}}$ *deterministic*
- $\mathcal{L}(\widehat{\mathcal{W}}) = \mathcal{L}(\mathcal{W})$

Using the ideal completion

Proposition

If X is an inductive invariant for $\mathcal{W}$,

then its ideal decomposition $\textsc{Idec}(X)\downarrow$

is a finitely-represented inductive invariant for $\widehat{\mathcal{W}}$.

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Property of being an inductive invariant carries over

Any set of the shape $\textsc{Idec}(Y){\downarrow}$ is finitely-represented in $\widehat{\mathcal{W}}$ $\square$

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Result in particular applies to $\mathsf{Cover} = \mathsf{Post}^*(I_1 \times I_2)\downarrow$.

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Result in particular applies to $\mathrm{Cover} = \mathrm{Post}^*(I_1 \times I_2)\downarrow$.

Remark: $\widehat{\mathcal{W}}$ is not necessarily a WSTS.

Conclusion

Regular separability for WSTS languages

Theorem

If two WSTS languages are disjoint,

one of them finitely branching or deterministic or ω^2,

then they are regularly separable.

Also in the paper...

1. A similar result for downward-compatible WSTS

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Theorem

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2. A size estimation for the case of Petri nets

Theorem

Given two Petri nets, their coverability languages can be separated by

- *Upper bound: an NFA of triply-exponential size*
- *Lower bound: a DFA of triply-exponential size*

Open problems

Expressibility results:

Are the inclusions strict?

$$\begin{aligned} \omega^2 - \text{WSTS languages} &\subseteq \text{det. WSTS languages} \\ \text{deterministic WSTS languages} &\subseteq \text{all WSTS languages} \end{aligned}$$

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Are disjoint WSTS languages always regularly separable?

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Separability results:
Are disjoint WSTS languages always regularly separable?

Crucial for both problems:
Expressiveness of infinitely-branching Rado WSTS

Thank you!

Questions?

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